Brane dynamics and 3D Seiberg duality on the domain walls of 4D $\mathcal{N} = 1$ SYM

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Brane dynamics and 3D Seiberg duality on the domain walls of 4D $\mathcal{N} = 1$ SYM

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Abstract: We study a three-dimensional U(k) Yang-Mills Chern-Simons theory with adjoint matter preserving two supersymmetries. According to Acharya and Vafa, this theory describes the low-energy worldvolume dynamics of BPS domain walls in four-dimensional $\mathcal{N} = 1$ SYM theory. We demonstrate how to obtain the same theory in a brane configuration of type IIB string theory that contains threebranes and fivebranes. A combination of string and field theory techniques allows us to re-formulate some of the well-known properties of $\mathcal{N} = 1$ SYM domain walls in a geometric language and to postulate a Seiberg-like duality for the Acharya-Vafa theory. In the process, we obtain new information about the dynamics of branes in setups that preserve two supersymmetries. Using similar methods we also study other $\mathcal{N} = 1$ CS theories with extra matter in the adjoint and fundamental representations of the gauge group.

Keywords: Supersymmetry and Duality, Brane Dynamics in Gauge Theories, Duality in Gauge Field Theories

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In honor of Mikhail Shifman,
on the occasion of his 60$^{\text{th}}$ birthday.

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1 Introduction

Brane configurations in string theory [1] have proven to be a useful tool in studying the strong coupling regime of gauge theories with various amounts of supersymmetry in diverse dimensions; see ref. [2] for a review. In this paper we study a certain class of three-dimensional Yang-Mills Chern-Simons (YM-CS) theories with two supersymmetries. Other YM-CS theories with larger amounts of supersymmetry attracted recently a lot of attention due to their relation with the worldvolume theory of M-theory membranes and the AdS_4/CFT_3 correspondence [3]. An interesting by-product of this discussion was the realization that some of these theories admit a Seiberg-type duality [4–7].

The theory that we mainly study here has a $U(k)$ gauge group and one $\mathcal{N} = 1$ adjoint scalar multiplet. It can be described as an $\mathcal{N} = 2$ super-Yang-Mills (SYM) theory with an $\mathcal{N} = 1$ Chern-Simons interaction that reduces the amount of supersymmetry by half. The action of the theory is

$$\mathcal{S} = \mathcal{S}_{\mathcal{N}=2 \text{ SYM}} + \mathcal{S}_{\mathcal{N}=1 \text{ CS}} , \tag{1.1}$$

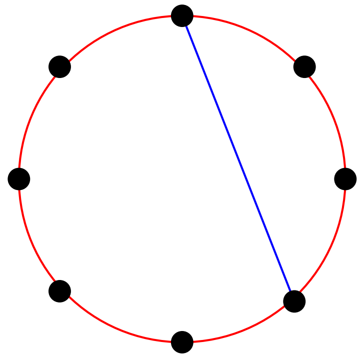

Figure 1. The vacua and k-walls of $\mathcal{N}=1$ SYM. The k-wall and the $(N-k)$-anti-wall interpolate between the same vacua. The present example is $N=8$: the depicted 3-wall is equivalent to a 5-anti-wall.

with

$$\mathcal{S}_{\mathcal{N}=2\ \mathrm{SYM}} = \frac{1}{4g^2_{\mathrm{YM}}}\int \mathrm{d}^3x\ \mathrm{Tr}\left((D\phi)^2 - F^2 + i\bar{\chi}\not{D}\chi + i\bar{\psi}\not{D}\psi + 2i\bar{\chi}[\phi,\psi]\right) \tag{1.2a}$$

$$\mathcal{S}_{\mathcal{N}=1\ \mathrm{CS}} = \frac{N}{4\pi}\int \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right) - \frac{N}{4\pi}\int \mathrm{d}^3x\ \bar{\chi}\chi\ . \tag{1.2b}$$

The gauge field A and the Majorana fermion χ form an $\mathcal{N}=1$ vector multiplet (F denotes the field strength of A). The real scalar field ϕ and the Majorana fermion ψ form an $\mathcal{N}=1$ scalar multiplet.

One motivation for studying this theory originates from the following observation. In $\mathcal{N}=1$ SYM theory in four dimensions, the $\mathrm{U}(1)_R$ symmetry is broken to $\mathbb{Z}_{2N}$ by the chiral anomaly. The $\mathbb{Z}_{2N}$ symmetry is further broken spontaneously down to $\mathbb{Z}_2$ by gaugino condensation. As a result, there are BPS domain walls which interpolate between the various vacua of the theory [8]. Witten proposed [9] that the domain walls of $\mathcal{N}=1$ SYM behave like D-branes for the QCD-string, as the QCD-string can end on them. In that case, one expects that there is a gauge theory living on the SYM walls. Later, it was argued by Acharya and Vafa (AV) [10], using string theory, that the theory that lives on k coincident domain walls of $\mathcal{N}=1$ SYM is the three-dimensional $\mathrm{U}(k)$ YM-CS gauge theory that appears in eqs. (1.1)–(1.2). Ref. [10] obtained this result from a large N transition in a type IIA setup that will be reviewed in subsection 2.4.

The appearance of the AV theory in this context raises the following question [11]: since the 'clockwise' interpolation between k vacua and the 'anti-clockwise' interpolation between $N-k$ vacua are the same in $\mathcal{N}=1$ SYM (see figure 1), is it sensible to conclude that the $\mathrm{U}(k)$ level N AV theory is equivalent to the $\mathrm{U}(N-k)$ level N theory? Furthermore, what is the nature of this duality in three-dimensional field theory terms?

In this work, we will answer this question by proposing that the $\mathrm{U}(k)$ and the $\mathrm{U}(N-k)$ theories form a pair of Seiberg dual theories. The two theories in this pair flow to the same infrared (IR) theory and hence naturally describe the same k-wall (or $(N-k)$-anti-wall) bound state. At low energies, below the energy scale set by the gauge boson mass (coming from the CS interaction), the standard Yang-Mills kinetic terms can be dropped and both theories become $\mathcal{N} = 1$ CS theories coupled to an $\mathcal{N} = 1$ scalar multiplet. We will argue for a duality that relates this pair of $\mathcal{N} = 1$ Chern-Simons-matter (CSM) theories. In the deep IR, where both theories become topological, the equivalence reduces to a well-known level-rank duality between bosonic Chern-Simons theories.

Additional arguments for Seiberg duality in three dimensions will be provided by using a brane configuration in type IIB string theory, which is an $\mathcal{N} = 1$ deformation of the setup that was used in [4] to argue for Seiberg duality in $\mathcal{N} = 2$ CSM theories. The brane configuration consists of k coincident D3-branes suspended between an NS5-brane and a $(1, N)$ fivebrane bound state (see section 2 for a detailed description). After T-duality this setup bears many similarities with the large-N dual string theory background of [10] but is not identical to it.

The study of this brane configuration will be doubly beneficial. On the one hand, it provides an intuitive geometric reformulation of non-perturbative gauge theory dynamics in three dimensions. On the other hand, we can use the field theory picture to learn more about brane dynamics in a setup that preserves only two supersymmetries. We will see, in particular, how the s-rule of brane dynamics and the brane creation (Hanany-Witten) effect work together with a perturbatively generated potential for a pseudo-modulus that binds suspended D3-branes into a bound state.

The main results of this paper are as follows. In section 3 we study the AV theory using a combination of string and field theory techniques and argue that it admits a Seiberg-like duality for $k \leqslant N$. When $k > N$, the s-rule dictates that supersymmetry is spontaneously broken in agreement with field theory expectations from the $\mathcal{N} = 1$ SYM theory in four dimensions. We evaluate the degeneracy $I_{k,N}$ of a k-wall by lifting the brane configuration to M-theory. The result is

$$I_{k,N} = \frac{N!}{k!(N-k)!}\,, \tag{1.3}$$

in agreement with the field theory calculation of ref. [10]. From the brane point of view, a particularly interesting part of the story is how D3-branes (corresponding to the domain walls of $\mathcal{N} = 1$ SYM) attract each other to form bound states with a tension given by the formula (3.1).

Two generalizations of the AV theory are discussed using brane techniques in section 4. The first generalization considers the addition of fundamental matter to the field content of the AV theory, and the second the addition of extra matter in the adjoint in the presence of a tree-level superpotential. We conclude in section 5 with a brief summary of our results and a list of interesting questions and open problems.

2 $\mathcal{N} = 1$ Chern-Simons-Matter theories from branes

Supersymmetric gauge theories in diverse dimensions arise naturally, as low-energy effective descriptions, in configurations of D-branes and NS5-branes in type II string theory (see [2] for a review). In this section we revisit a configuration that realizes Chern-Simons-Matter theories with $\mathcal{N} = 1$ supersymmetry. For special values of the parameters that characterize the configuration we recover the AV CS theory which describes the low-energy dynamics on the domain walls of $\mathcal{N} = 1$ SYM theory in four dimensions.

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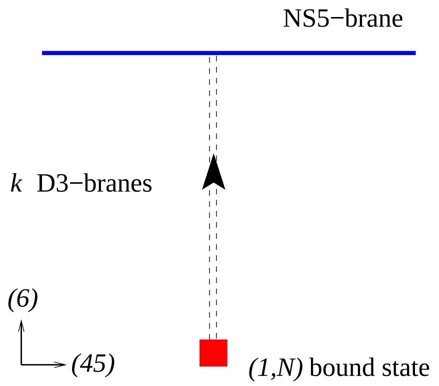


k D3-branes	: 0 1 2				$\lvert 6 \rvert$
NS5-brane	: 0 1 2	3	4	5	
$(1,N)$ fivebrane	: 0 1 2	$\left[{3 \atop 7}\right]_{\theta+\varphi+\psi}$	$\left[{4 \atop 8}\right]_{\psi}$	$\left[{5 \atop 9}\right]_{\varphi}$	

Figure 2. The brane configuration of interest. For generic angles ψ, φ only two supercharges are conserved. The low-energy dynamics is described by a three-dimensional $\mathcal{N}=1$ CSM theory.

2.1 The brane setup of interest

Consider a configuration of branes consisting of k D3-branes, one NS5-brane and a bound state of one NS5-brane and N D5-branes, *i.e.* a $(1,N)$ fivebrane, oriented as depicted in figure 2. The D3-branes are suspended between the fivebranes and have a finite extent L along the x^6 direction, a feature captured by the notation $|6|$. The orientation of the $(1,N)$ bound state along the (37), (48) and (59) planes is given by the angles θ, ψ and φ.[1] The configuration preserves at least two supersymmetries for generic angles φ, ψ provided θ obeys the following relation [12, 13]:

$$\tan\theta = g_s N\,, \tag{2.1}$$

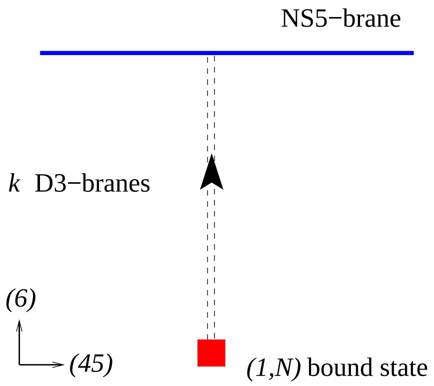
where g_s is the string coupling.

The low-energy effective theory that describes the dynamics of this configuration is a field theory that lives on the k D3-branes. At energies below the Kaluza-Klein (KK) scale $m_{\rm KK} = \frac{1}{L}$ the effective theory is three-dimensional. In the presence of the $(1,N)$ fivebrane it is known [13] that this theory is a U(k) YM-CS theory at level N coupled to matter. The matter consists of three $\mathcal{N}=1$ real scalar multiplets in the adjoint of U(k), associated with the directions x^3, x^4 and x^5; we will denote them as Φ_3, Φ_4 and Φ_5, respectively. For generic angles ψ and φ these multiplets have independent masses and the field theory is an $\mathcal{N}=1$ YM-CSM theory.

The three-dimensional gauge field A is part of the $\mathcal{N}=1$ vector multiplet which in addition includes the gaugino Majorana fermion χ. The scalar multiplets Φ_i include a real scalar ϕ_i and a Majorana fermion ψ_i $(i=3,4,5)$. The low-energy effective action of these fields will be discussed in more detail in a moment.

The $\mathcal{N}=1$ supersymmetry is enhanced to $\mathcal{N}=2$ or $\mathcal{N}=3$ for special values of the angles ψ, φ. Enhancement to $\mathcal{N}=2$ occurs when $\psi=-\varphi$. The special case $\psi=\frac{\pi}{2}$ was the main focus of the recent work [4] that formulated a Seiberg-like duality for $\mathcal{N}=2$ CSM theories. An extra set of N_f D5-branes oriented along the directions (012789) was also present in that setup. Similar D5-branes and their implications for $\mathcal{N}=1$ CSM dynamics will be discussed in subsection 4.1.

[1] $\left[{i \atop j}\right]_{\vartheta}$ denotes that the brane is oriented along the (ij) plane at an angle ϑ with respect to the axis i.



k D3-branes	: 0 1 2			$\lvert 6 \rvert$	
NS5-brane	: 0 1 2	3	4 5		
$(1,N)$ fivebrane	: 0 1 2	3	$\left[{}^{5}_{9}\right]_{-\frac{\pi}{2}-\theta}$		8

Figure 3. The special configuration that realizes the Acharya-Vafa CS theory. Compared to the generic configuration in figure 2 there is now an extra common direction (x^3) between the fivebranes.

Further enhancement of the supersymmetry to $\mathcal{N}=3$ occurs when all three angles are correlated: $\psi=-\varphi=\theta$. A similar setup, with the x^6 direction compactified, was crucial in the recent discussion of low-energy descriptions of the M2-brane worldvolume dynamics based on CSM theories [3, 14].

One of the early motivations for studying the generic $\mathcal{N}=1$ setup in figure 2 was to formulate the conditions for spontaneous breaking of supersymmetry in Chern-Simons theories as a consequence of brane dynamics [15–17]. These conditions will play an important role in the next section.

2.2 Getting the AV field theory

The AV theory is a U(k) $\mathcal{N}=1$ CS theory at level N coupled to a classically massless $\mathcal{N}=1$ adjoint scalar multiplet. To recover this theory from the brane configuration in figure 2 we must tune the angles ψ, φ in such a way that one of the scalar multiplets Φ_i becomes massless and the remaining two extremely massive. This can be achieved, for example, by setting

$$\psi=\frac{\pi}{2}\,, \qquad \varphi=-\frac{\pi}{2}-\theta\,, \tag{2.2}$$

giving rise to a brane configuration with branes oriented as summarized in figure 3. Φ_3 is now a massless scalar multiplet, Φ_4 is infinitely massive and Φ_5 is massive with a mass m_5 that will be discussed in a moment.

Notice that by setting $N=0$ we replace the fivebrane bound state with an NS5-brane which, according to equation (2.1), is oriented along the directions (012389). This configuration preserves four supersymmetries and gives rise to the three-dimensional $\mathcal{N}=2$ SYM theory on the D3-branes. With a non-zero value of N we expect to have an extra $\mathcal{N}=1$ Chern-Simons interaction at level N. As reviewed in the introduction, this is the theory that according to [10] describes the infrared dynamics on the domain walls of $\mathcal{N}=1$ SYM in four dimensions.

Before establishing this fact we have to tie a loose end. The low-energy field theory on the D3-branes includes, for non-zero N, the extra massive scalar multiplet Φ_5. The mass of this multiplet is [17, 18]

$$m_5 = \frac{\cot\,\theta}{L} = \frac{m_{\rm KK}}{g_s N} = \left(\frac{m_{\rm KK}}{m_{\rm CS}}\right)^2 m_{\rm CS}\,, \tag{2.3}$$

where $m_{\rm CS}$ is the Chern-Simons induced mass of the gauge field

$$m_{\rm CS} = g_{\rm YM}^2 N\,, \tag{2.4}$$

and $g_{\rm YM}$ the dimensionful three-dimensional Yang-Mills coupling given by the equation

$$\frac{1}{g_{\rm YM}^2} = \frac{L}{g_s}\,. \tag{2.5}$$

In the perturbative string regime of interest,

$$\frac{m_{\rm KK}}{m_{\rm CS}} = \frac{1}{g_s N} \gg 1\,, \tag{2.6}$$

the KK modes are very heavy and can be ignored. Then the hierarchy of scales

$$m_{\rm CS} \ll m_{\rm KK} \ll m_5 \tag{2.7}$$

guarantees that the multiplet Φ_5 can be safely integrated out.

To obtain the precise Lagrangian of the low-energy theory on the brane setup of figure 3, we can start from the U(k) $\mathcal{N} = 3$ CS theory at level N, whose Lagrangian is completely fixed by supersymmetry. It describes the low-energy dynamics of the brane configuration with $\psi = -\varphi = \theta$. In components, the action of this theory reads [16, 19]

$$\begin{aligned}
\mathcal{S} &= \mathcal{S}_{\rm YM} + \mathcal{S}_{\rm CS} \\
&= \frac{1}{4g_{\rm YM}^2}\int {\rm d}^3x\ {\rm Tr}\Big\{-F^2 + \sum_{i=3}^{5}\left[(D\phi_i)^2 + C_i^2 + i\bar\psi_i \not{D}\psi_i\right] + i\bar\chi \not{D}\chi + i\bar\chi \\
&\qquad\qquad + [\psi_3,\phi_3] + i\bar\chi[\psi_5,\phi_5] + i\bar\psi_3[\chi,\phi_3] + i\bar\psi_3[\psi_5,\phi_4] + i\bar\psi_5[\chi,\phi_5] \\
&\qquad\qquad + i\bar\psi_5[\psi_3,\phi_4] - 2i\bar\chi[\psi_4,\phi_4] - 2i\bar\psi_3[\psi_4,\phi_5] + \frac{1}{2}\sum_{i<j}[\phi_i,\phi_j]^2\Big\} \\
&\quad + \frac{N}{4\pi}\int {\rm Tr}\left(A\wedge {\rm d}A + \frac{2}{3}A^3\right) \\
&\quad + \frac{N}{4\pi}\int {\rm d}^3x\left\{-\bar\chi\chi + \sum_{i=3}^{5}\left[(-)^i\bar\psi_i\psi_i + 2\phi_i C_i\right] + \frac{1}{3}\sum_{ijk}\epsilon_{ijk}\phi_i[\phi_j,\phi_k]\right\}\,.
\end{aligned} \tag{2.8}$$

The C_i ($i = 3, 4, 5$) are auxiliary scalars in the Φ_i multiplets.

All the fields in the above action have the same mass whose value is fixed by the $\mathcal{N} = 3$ CS term in the last line of (2.8). Configurations with less supersymmetry can be obtained by changing the mass of each of the $\mathcal{N} = 1$ scalar multiplets separately. In particular, by tuning the bare mass of Φ_3, Φ_4 and Φ_5 to zero, infinity and m_5, respectively, integrating

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out Φ_4 and Φ_5 and renaming $\phi_3 = \phi$, $\psi_3 = \psi$, we obtain the AV action, as described in the introduction,

$$\begin{aligned}\mathcal{S}_{\rm AV} = & \frac{1}{4g_{\rm YM}^2}\int {\rm d}^3x\ {\rm Tr}\left(-F^2 + (D\phi)^2 + i\bar{\psi}\not{D}\psi + i\bar{\chi}\not{D}\chi + 2i\bar{\chi}[\phi,\psi]\right) \\ & + \frac{N}{4\pi}\int {\rm Tr}\left(A\wedge {\rm d}A + \frac{2}{3}A^3\right) - \frac{N}{4\pi}\int {\rm d}^3x\,\bar{\chi}\chi\ .\end{aligned} \tag{2.9}$$

2.3 Comments on the dynamics of the AV theory

At low energies (below $m_{\rm CS}$) the standard kinetic term of the gauge field and the kinetic term of the gaugino χ can be dropped and the YM-CS action for the AV theory (2.9) becomes the action of $\mathcal{N}=1$ CS theory coupled to a massless $\mathcal{N}=1$ scalar multiplet,

$$\begin{aligned}\mathcal{S}_{\rm AV\text{-}CSM} = & \frac{N}{4\pi}\int {\rm Tr}\left(A\wedge {\rm d}A + \frac{2}{3}A^3\right) - \frac{N}{4\pi}\int {\rm d}^3x\,\bar{\chi}\chi \\ & + \frac{1}{4g_{\rm YM}^2}\int {\rm d}^3x\ {\rm Tr}\left\{(D\phi)^2 + i\bar{\psi}\not{D}\psi + 2i\bar{\chi}[\phi,\psi]\right\}\ .\end{aligned} \tag{2.10}$$

The massive gaugino can be integrated out to obtain the classically marginal quartic interaction of the form $[\phi,\psi][\phi,\bar{\psi}]$.

$\mathcal{N}=1$ supersymmetry is not enough to guarantee the absence of quantum corrections to this action. In a similar situation with $\mathcal{N}=2$ supersymmetry, *e.g.* the situation of $\mathcal{N}=2$ CS theory coupled to an $\mathcal{N}=2$ chiral multiplet in the adjoint without superpotential interactions, it is known [20] that quantum effects do not generate any relevant interactions and hence the theory is an exact CFT. With just $\mathcal{N}=1$ supersymmetry relevant interactions can and will be generated.

Indeed, one can show explicitly in the AV theory [11, 21] that ϕ is not a true modulus and that quantum corrections lift the classical moduli space parametrized by the vacuum expectation values (VEVs) of gauge-invariant polynomials in ϕ. The lifting is a $1/N$ effect. The reason is that in the large N limit the k-wall becomes a collection of k non-interacting fundamental domain walls that can be separated freely. This is in agreement with our brane picture where in the limit $N\to\infty$, $\theta\to\pi/2$ and the theory acquires a quantum moduli space.

One can split the scalar multiplet as $\Phi = \Phi_0 + \hat{\Phi}$ according to the decomposition $u(k) \simeq u(1)\oplus su(k)$. Performing the calculation of a two-loop Coleman-Weinberg effective potential in the Coulomb branch of the U(2) AV theory, one finds a potential of the form [11]

$$V(u) \sim \frac{1}{N}\frac{u}{1+u}\,, \tag{2.11}$$

where $u \equiv \langle {\rm Tr}(\hat{\phi}^2)\rangle/m_{\rm CS}^2$. This perturbative result captures the leading $1/N$ effects. Higher order corrections are expected to modify the potential (2.11). In general, an attractive potential is generated near the origin for u but no potential is generated for the overall 'center-of-mass' VEV $\langle\phi_0\rangle$. We will re-encounter these quantum effects in the next section where we discuss the brane dynamics in the type IIB string theory setup of figure 3.

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We observe that the leading term of the potential (2.11) is a quadratic term with mass

$$m_{\text{LOOP}} = \frac{m_{\text{CS}}}{N}\,, \tag{2.12}$$

which is parametrically smaller than the CS mass m_{CS} in the large-N limit. Hence, at energies below m_{LOOP}, the U(k) CSM theory becomes a topological field theory — the bosonic CS theory — with an additional *decoupled* free massless real scalar field ϕ_0 and its superpartner. With the exception of the decoupled massless scalar multiplet we would have obtained the same infrared dynamics for any of the low-energy theories that live on the D3-branes of the general setup in figure 2, where a mass is present for the scalar multiplets already in the tree-level Lagrangian.

2.4 T-duality and the AV string theory setup

The brane system in type IIB string theory that appears in figure 3 is related to the Acharya-Vafa setup in type IIA string theory, but exhibits some differences. To see the relation, one may compactify x^7 on a circle and T-dualize along this direction. Before analyzing this transformation it will be useful to recall the setup of Acharya and Vafa in type IIA string theory.

The starting point is type IIA string theory on $\mathbb{R}^{3,1}$ times the deformed conifold. Wrapping N D6-branes around the non-vanishing three-cycle one obtains at low energies on the D6-branes four-dimensional $\mathcal{N} = 1$ SYM. It has been argued [22] that there is a large-N holographic description of this theory which involves a geometric transition from the deformed conifold to a resolved conifold. The D6-branes disappear in the resolved conifold and get replaced by their RR flux going through the non-vanishing two-cycle of the blown-up singularity. A string propagating in this background can be interpreted as the QCD-string of $\mathcal{N} = 1$ SYM. D4-branes wrapping the non-vanishing two-cycle of the resolved conifold are interpreted as domain walls in the $\mathcal{N} = 1$ SYM theory.

Now let us return to our setup of fivebranes. Consider first the case with $N = 0$. Then the setup in figure 3 consists of two NS5-branes, respectively along (012345) and (012389), and k D3-branes along (012|6|). T-dualizing along x^7 transforms the system of two fivebranes into the resolved conifold, whose blow-up parameter is controlled by L [23, 24]. This type IIA setup is identical to the setup of ref. [22] after the large N transition. The D3-branes are mapped to wrapped D4-branes and there is no RR flux through the two-cycle.[2]

Consider now what happens when we replace the second NS5-brane by the $(1, N)$ bound state that has a modified orientation along $(01238\,\left[{5 \atop 9}\right]_{-\frac{\pi}{2}-\theta})$. After the same T-duality along x^7 one gets a resolved conifold with RR flux and D4-branes around the two-cycle. More specifically, the NS5-brane turns into a Kaluza-Klein (KK) monopole stretched in the directions (012345), whose charge is associated with the T-dual of x^7, and the $(1, N)$ bound state turns into a U-dual of the KK dyon of [26], namely, an RR flux on a (differently oriented) KK monopole. Combining the KK monopole and dyon gives rise to

[2]The T-dual of Vafa's setup before the transition is given by a 'brane box' model [25] with N D5-branes filling a disc bounded by an NS5-brane.

a resolved conifold with N units of RR flux. However, the RR flux is oriented in different directions compared to the setup of ref. [22].

Despite the differences between the T-dual of our setup and the setup of [22], we have seen that the IR theory on the k D-branes is the same in both cases and describes the low-energy dynamics of the domain wall that interpolates between the ℓ-th and the $(\ell + k)$-th vacuum in $\mathcal{N} = 1$ SYM.

3 D-brane dynamics and domain walls in 4D SYM

Having established a relation between the brane configuration in figure 3 and the four-dimensional $\mathcal{N} = 1$ SYM theory we now proceed to explore how known facts in one theory map to known facts in the other. We will find that some properties are easy to establish in one formulation and difficult in the other making this comparison a fruitful exercise for both theories.

3.1 Pseudo-moduli, the s-rule and brane creation

We have argued that the $\mathcal{N} = 1$ CS theory on k suspended D3-branes is coupled to an $\mathcal{N} = 1$ chiral multiplet Φ in the adjoint of the U(k) gauge group. The vacuum expectation values of gauge-invariant polynomials in the scalar field ϕ parametrize the positions of the D3-branes in the transverse direction x^3. Classically, there is no potential for them and the three-branes can move apart without any cost of energy.

In brane setups with more supersymmetry ($\mathcal{N} = 2$ and higher) similar (complex) moduli exist both classically and quantum mechanically. With just $\mathcal{N} = 1$ supersymmetry, however, quantum corrections lift the classical moduli space. As we reviewed in subsection 2.3, in field theory these effects generate a Coleman-Weinberg potential for the VEVs associated with the SU(k) adjoint scalar $\hat{\phi}$ (see eq. (2.11) for the $k = 2$ case, *i.e.* for two domain walls) stabilizing them at the origin. The U(1) part $\langle\phi_0\rangle$ remains a modulus which is consistent with the expectation that we can arbitrarily place the center of mass of k D3-branes at any point along the x^3 direction.

Since there are no tachyons along the classical moduli space a phase transition is not anticipated as we go from the field theory regime to the perturbative brane regime. Consequently, a similar stabilization of the pseudo-modulus at the origin is expected also in the perturbative brane regime. We will present an alternative indirect argument for this in a moment. For two or more D3-branes stretching between the fivebranes in figure 3 the stabilization of the pseudo-modulus implies that while we can freely move their center of mass along the x^3 direction, we cannot separate them without some cost of energy. An attractive force between the D3-branes forms a bound state that behaves as a single object.

The formation of D3-brane bound states in our setup, labelled by k, matches nicely what is expected from domain walls in the $\mathcal{N} = 1$ SYM theory. The bound states have a non-trivial tension which is a certain function of the parameters k and N. The form of this function will be discussed shortly. Now we want to discuss the precise range of the rank k of the gauge group.

Supersymmetry restricts the number k of D3-branes that can be suspended between the fivebranes in the brane setup of figure 3. The standard s-rule of brane dynamics continues to hold in our case and dictates that the configuration is supersymmetric if and only if $k \leqslant N$. It is particularly interesting to pinpoint the ingredients that conspire to make the s-rule work in our setup.

A standard argument for the validity of the s-rule is the following. By moving the $(1, N)$ bound state along the x^6 direction past the NS5-brane k D3-branes are carried along and become anti-D3-branes. During the crossing of the fivebranes N D3-branes are created via the brane creation effect [1]. The brane creation effect ensures that the dynamics is smooth during the crossing and that the amount of supersymmetry of the original configuration is preserved in the final configuration. Hence, in accordance with the s-rule of the original setup, for $k \leqslant N$ the annihilation of k brane/anti-brane pairs leaves behind a supersymmetric configuration of $N - k$ suspended D3-branes. In the opposite regime ($k > N$) the annihilation of N brane/anti-brane pairs leaves behind a non-supersymmetric configuration of $k - N$ suspended anti-D3-branes.

In the absence of the attractive potential between the D3-branes a contradiction with the s-rule would have been obtained. We would have been able to freely separate the D3-branes along the x^3 direction to obtain a supersymmetric configuration for any k. Hence, the validity of the s-rule requires the presence of the attractive potential between the D3-branes in the brane regime and confirms our expectations from field theory [11, 21].

The Witten index corroborates this picture. The validity of the s-rule requires that the Witten index is non-zero for $k \leqslant N$ in field theory and zero in the opposite regime. Indeed, for the AV CS theory the Witten index was computed in [10] and was found to be proportional to $\frac{1}{(N-k)!}$. At the same time, the independence of the Witten index from the mass of $\hat{\Phi}$ [10] fits nicely with the fact that the standard s-rule holds for general angles in the brane setup of figure 2, where Φ is massive. In the next subsection we will further show how branes provide a natural geometric interpretation of the precise value of the Witten index for arbitrary values of k.

Since there is an upper bound on the number k of D3-branes that can be stretched between fivebranes in our setup without breaking supersymmetry we conclude that there are only N distinct supersymmetric D3-brane bound states for $k = 1, 2, \cdots, N$. Later we will see that the N-th state with $k = N$ is equivalent to the pure vacuum $k = 0$ state, hence the number of distinct D3-brane bound states obtained in this way is actually $N - 1$. This number matches exactly the number of different domain walls in the four-dimensional $\mathcal{N} = 1$ SYM theory and is further evidence for the validity of the above picture.

3.2 Witten index and the degeneracy of domain walls

As we reviewed in the introduction, the k-th domain wall in the $\mathcal{N} = 1$ SYM theory has a degeneracy given by the index $I_{k,N}$, presented in eq. (1.3). In the CS theory that captures the low-energy dynamics of the k-th domain wall this index counts the number of supersymmetric vacua. We can ask whether this degeneracy is visible in the brane setup of figure 3.

At first sight we seem to get a different answer. k D3-branes stretch between an NS5-brane and a $(1, N)$ fivebrane in an apparently unique way. A similar mismatch between (1.3) and the counting of BPS branes in string theory was also observed in the type

IIA context of ref. [10] and earlier in the context of MQCD in [27]. Ref. [10] pointed out that this issue is related to the global boundary conditions on the domain walls. Counting vacua in Minkowski space is different from counting vacua in a toroidally compactified theory. The computation of the Witten index in the toroidally compactified AV CSM theory gives the expected answer (1.3) [10]. Hence, we can now refine the above question. Can we reproduce the index (1.3) in the brane setup of figure 3 after compactifying the worldvolume of the D3-branes?

The answer, which has a simple geometric interpretation, lies already in a work by Ohta [17]. We will review the argument here with the appropriate modifications. Consider compactifying one of the worldvolume directions of the D3-branes, say the direction x^2. T-dualizing the setup along x^2, adding the M-theory circle x^{10} and lifting to M-theory we obtain a configuration with M2-branes along $(01|6|)$ stretched between an M5-brane along (012345) and an M5-brane along $(0138\left[{}^5_9\right]_{-\frac{\pi}{2}-\theta})$ wrapping the cycle $\alpha_2 + N\alpha_{10}$, where α_2 is the cycle associated with the direction x^2 and α_{10} the cycle associated with the direction x^{10}. The M5-branes intersect N times on the $(2,10)$ torus. An M2-brane stretching between them without breaking the supersymmetry is necessarily attached on each of these N intersection points. According to the s-rule no more than one M2-brane can be attached to the same intersection point if we require supersymmetry. Hence, the counting of possible supersymmetric configurations of stretched M2-branes boils down to a counting of k M2-branes distributed along the N M5 intersection points without violating the s-rule. For $k \leqslant N$ the answer is trivially given by eq. (1.3). For $k > N$ the s-rule is necessarily violated and no supersymmetric vacuum exists. This picture provides a simple, intuitive understanding of the Witten index (1.3) in string/M theory.

3.3 Tension formula

Another characteristic feature of the BPS domain walls in $\mathcal{N} = 1$ SYM theory is their tension, given by the sine formula [8],

$$T_k = \frac{N^2\Lambda^3}{4\pi^2} \sin\frac{\pi k}{N}\,, \tag{3.1}$$

for the k-th domain wall, in terms of the dynamically generated QCD scale Λ.

Because of the attractive potential between the D3-branes, the k-th domain wall maps to a bound state of k D3-branes in our setup. The attractive force is weak for $\frac{k}{N} \ll 1$ and to leading order in $\frac{k}{N}$ the tension of k suspended D3-branes is

$$T_k = kT_1 + \mathcal{O}(N^{-1}) \sim \frac{L}{g_s \ell_s^4} k + \mathcal{O}(N^{-1})\,. \tag{3.2}$$

Identifying

$$g_s \sim \frac{L}{N\ell_s}\,, \qquad \ell_s \sim \frac{1}{\Lambda}\,, \tag{3.3}$$

we reproduce (up to a numerical coefficient) the leading order term of T_k in (3.1). The identification (3.3) assumes parameters typical of large-N string theory duals of a QFT (see e.g. [11, 22]). Hence, the string coupling g_s is proportional to $\frac{1}{N}$, and the string scale ℓ_s is set by the QCD string tension. Notice that in this identification $m_{\rm CS} \sim \Lambda$.

Reproducing the full k/N dependence of T_k requires a difficult non-perturbative computation in CS theory. A perturbative treatment of the tension in the AV CS-YM theory can be found in [11, 21]. From the point of view of the type IIB setup in figure 3 an exact computation of T_k requires a detailed knowledge of the forces that form the D3-brane bound states, which is currently lacking. It is interesting, however, that by turning the tables around and viewing the D3-branes as domain walls in the four-dimensional $\mathcal{N}=1$ SYM theory we can determine exactly the k D3-branes tension as in (3.1).

We conclude by noticing the identity

$$T_k = T_{N-k} \ . \tag{3.4}$$

As we will review in a moment this identity has a natural explanation in the $\mathcal{N}=1$ SYM theory. We will be able to recover it independently in our brane setup together with a statement of Seiberg duality for the $\mathcal{N}=1$ AV CSM theory.

3.4 Seiberg duality

We are now coming to a different aspect of the dynamics of the $\mathcal{N}=1$ AV CSM theory: a Seiberg-like duality that relates the U(k) description to a U($N-k$) one.

Let us first review briefly a related example with $\mathcal{N}=2$ supersymmetry that appeared in [5, 6]. It involves the U(k) $\mathcal{N}=2$ CS theory at level N coupled to an $\mathcal{N}=2$ chiral multiplet X in the adjoint representation of the gauge group. To formulate Seiberg duality in this case a tree-level superpotential is necessary,

$$W_{n+1} = g_n \, \mathrm{Tr}\, X^{n+1} \, , \quad n \geq 1 \ . \tag{3.5}$$

The dual is a level N U($nN-k$) CS theory with an adjoint and a superpotential (3.5). In the special case of $n=1$ the superfield X is massive and can be integrated out to recover the $\mathcal{N}=2$ CS theory which is a topological field theory. By a standard argument that will be reviewed in a moment, duality in CS theory reduces, in this case, to level-rank duality in an SU(N) WZW model.

The situation in the AV theory is similar, but instead of an $\mathcal{N}=2$ CS theory coupled to an $\mathcal{N}=2$ chiral multiplet in the adjoint we have an $\mathcal{N}=1$ CS theory coupled to an $\mathcal{N}=1$ adjoint multiplet. A tree-level potential is absent, but one is generated at the quantum level. In that sense, the AV theory is similar to the $n=1$ special case of (3.5). A Seiberg duality between the U(k) AV theory and the U($N-k$) AV theory (both at level N) is anticipated.

In the deep IR (below the energy scale m_{LOOP} set by the loop corrections – see eq. (2.12)) one is left with the bosonic U(k) CS theory plus a decoupled free massless real scalar field and a fermion. Integrating out the massive fermions gives rise to a shift of the level from N to $N-k$ [28] (the contribution of the gauge field is not included here). Hence, the IR theory includes a level $N-k$ SU(k) pure CS theory. In the infrared, the rank of this dual theory is similarly shifted from N to k, giving rise at low energies to an SU($N-k$) pure CS theory at level k. The SU(k) level $N-k$ and the SU($N-k$) level k theories are equivalent by the CS-WZW correspondence of [29] and level-rank duality in SU(N) WZW models. In this way, we recover a duality between the U(k) and U($N-k$) AV theories in the infrared.

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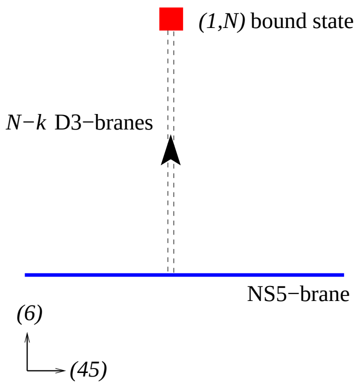

$$\begin{array}{lll} N-k\,\text{D3-branes} & : 0\,1\,2 & \qquad\qquad |6| \\ \text{NS5-brane} & : 0\,1\,2\;\;3\;\;4\;\;5 & \\ (1,N)\ \text{fivebrane} & : 0\,1\,2\;\;3\;\;\left[{}^{5}_{9}\right]_{-\frac{\pi}{2}-\theta} & \qquad\qquad 8 \end{array}$$

Figure 4. The brane setup that realizes the dual Acharya-Vafa CS theory.

Further aspects of this duality can be deduced by reformulating it in the context of string theory and in the context of the four-dimensional $\mathcal{N}=1$ SYM theory, as we discuss next.

Seiberg duality from brane dynamics. As we mentioned above, in the type IIB setup of figure 3 no phase transitions are anticipated as we change continuously the separation L of the NS5-brane and the $(1,N)$ bound state along the x^6 direction. Even when we pass the $(1,N)$ fivebrane through the NS5-brane the process is smooth and the IR dynamics on the worldvolume of the D3-brane remains invariant.

We have already observed that by passing the $(1,N)$ fivebrane through the NS5-brane one is left with the configuration of figure 4 that involves $N-k$ suspended D3-branes. The new configuration realizes a dual description of the original $U(k)$ AV CS theory. Below the energy scale set by m_{CS} this description is still an $\mathcal{N}=1$ CSM theory at level N coupled to a massless adjoint multiplet with a dual gauge group $U(N-k)$. In the dual description the supersymmetry conserving condition $k \leqslant N$ becomes a classical fact related to the rank of the gauge group in the dual theory. This is typical in dualities of this kind.

Seiberg duality as a charge conjugation symmetry. The embedding of the AV theory into the four-dimensional $\mathcal{N}=1$ SYM theory provides yet another way to look at this duality.

From the $\mathcal{N}=1$ SYM point of view the $U(k)$ AV CS theory at level N describes the IR dynamics of the theory that lives on the k-th domain wall which interpolates 'clockwise' between the j-th and the $(j+k)$-th vacuum. By charge conjugation symmetry this domain wall is equivalent to the $(N-k)$-th anti-wall that interpolates 'anti-clockwise' between the $(j+N-k)$-th and the j-th vacuum. The IR physics of the theory that lives on this anti-wall is captured by the $U(N-k)$ dual AV CS theory. We observe that in this case Seiberg duality, which is a hard non-perturbative statement in three-dimensions, becomes a statement that follows directly from a simple symmetry, *i.e.* charge conjugation symmetry, in the 'parent' four-dimensional gauge theory. It would be interesting to know if there are other examples in field theory where Seiberg duality can be derived in this way.

The duality between the U(k) and the U($N-k$) $\mathcal{N}=1$ AV theories is a strong/weak coupling duality. The 't Hooft coupling of the U(k) theory (2.10) is $\lambda = \frac{k}{N}$ whereas that of the U($N-k$) theory is

$$\widetilde{\lambda} = \frac{N-k}{N} = 1-\lambda \ . \tag{3.6}$$

The strongly coupled point at $k=\frac{N}{2}$ (for N even) is self-dual under the duality. It would be interesting to know if the theory enjoys special properties at this point.

At low energies below $m_{\rm LOOP}$, where the theory is topological, the duality is, as we mentioned, a consequence of level-rank duality in WZW models. The observables of the bosonic CS theory are Wilson loop operators [29]. Charge conjugation symmetry in the context of $\mathcal{N}=1$ SYM theory predicts that the expectation values of these Wilson loop operators are invariant under the replacement $k \to N-k$. This is consistent with level-rank duality.

In addition, the D-brane and domain wall perspectives from string theory and the $\mathcal{N}=1$ SYM theory, respectively, suggest that the duality extends beyond the topological data of the IR theory below $m_{\rm LOOP}$. Note that the tension of the domain walls can be calculated using the AV theory. Specifically, when N is large $T_k = kT_1 + V$, where V is the binding energy. The binding energy is calculated by a Coleman-Weinberg potential. The tension, which is *not* a topological datum, admits $T_k = T_{N-k}$, namely a Seiberg dual relation. Therefore, it suggests that the equivalence goes beyond the topological data and is valid throughout the whole RG flow from the scale $m_{\rm CS}$ to the far IR.

4 Generalizations

The type IIB brane construction suggests a number of interesting generalizations of the $\mathcal{N}=1$ AV CSM theory. Two of them will be discussed briefly in what follows. In the first example we consider extra matter in the fundamental representation of the gauge group. The second example, which includes additional matter in the adjoint, is a particularly interesting case where one can argue, in a certain regime of parameters, that the infrared theory is an interacting conformal field theory, instead of a topological field theory.

4.1 Adding flavor

Adding N_f D5-branes, oriented along the directions (012789), in the brane setup of figure 3 we obtain the configuration of figure 5(a). In this configuration the low-energy theory on the D3-branes becomes a U(k) $\mathcal{N}=1$ CS-YM theory at level N coupled to a massless $\mathcal{N}=1$ scalar multiplet in the adjoint and N_f pairs of $\mathcal{N}=2$ chiral multiplets Q^i, $\widetilde{Q}_i$ ($i=1,\cdots,N_f$) in the fundamental and anti-fundamental representations of the gauge group.[3]

The Lagrangian that describes the low-energy dynamics of D3-branes in the setup that appears in figure 5(a) includes: (i) the $\mathcal{N}=1$ AV CS-YM Lagrangian (1.1)–(1.2) for the $\mathcal{N}=1$ vector multiplet and the adjoint superfield Φ_3, (ii) the standard $\mathcal{N}=2$ kinetic terms

[3]One can also consider adding N_f D5-branes in the more general brane setup of figure 2. In that case, the $\mathcal{N}=1$ scalar multiplet is massive and comes along with two additional massive $\mathcal{N}=1$ scalar multiplets. The resulting setup, which will not be discussed explicitly here, bears many similarities with the $\mathcal{N}=2,3$ setups analyzed in [4].

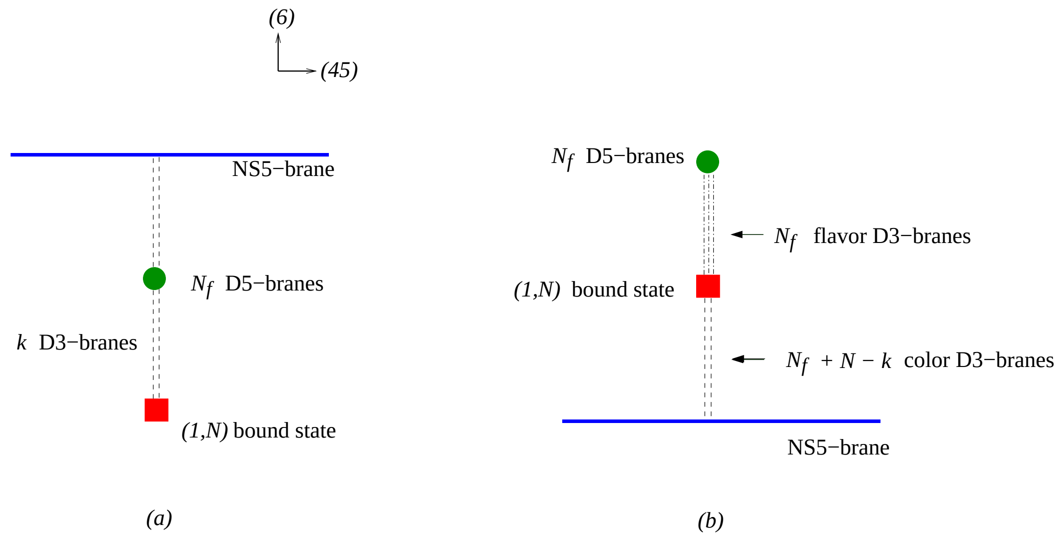

Figure 5. Figure (a) depicts the brane setup that realizes a flavored version of the U(k) Acharya-Vafa CS theory. The setup in figure (b) realizes its U($N_f + N - k$) dual. The orientation of the D3-branes, the NS5-brane and the fivebrane bound state is the same as that in figure 3. The D5-branes are oriented along the directions (012789).

for the quark multiplets Q^i, $\widetilde{Q}_i$, and (iii) a tree-level superpotential coupling between the multiplets Φ_3 and Q^i, $\widetilde{Q}_i$,

$$\int d^2\theta \; Q^i \Phi_3 \widetilde{Q}_i + c.c. \,, \tag{4.1}$$

written here in the $\mathcal{N} = 2$ formalism with Φ_3 regarded as the real part of an $\mathcal{N} = 2$ chiral superfield.

The infrared dynamics of this theory is controlled by several factors. The cubic coupling (4.1) is classically relevant and affects the RG flow, but there are additional interactions generated by loop effects. In particular, there is no symmetry preventing the generation of masses for the matter fields. At weak coupling, one expects that the quantum generated mass terms take over. In the deep infrared this leads again to a description in terms of the topological CS theory. It is unclear if a non-trivial interacting fixed point can arise at large values of the effective coupling k/N.

From the string theory embedding we can read off immediately the following properties. First, the s-rule of brane dynamics dictates that the theory exhibits spontaneous breaking of supersymmetry when

$$k > N_f + N \;. \tag{4.2}$$

Second, by passing the N_f D5-branes and the $(1, N)$ bound state through the NS5-brane along x^6 we obtain the configuration in figure 5(b) that realizes a dual U$(N_f + N - k)$ $\mathcal{N} = 1$ CSM theory at level N coupled to the following matter multiplets: (i) an $\mathcal{N} = 1$ scalar multiplet $\widetilde{\Phi}_3$, (ii) N_f pairs of dual $\mathcal{N} = 2$ quark multiplets q_i, $\tilde{q}^i$, and (iii) a set of gauge-singlet dual meson $\mathcal{N} = 1$ scalar multiplets M_i^j $(i, j = 1, \cdots, N_f)$. The scalar component of the $\widetilde{\Phi}_3$ multiplet describes the fluctuations of the $N-k$ color D3-branes in the

x^3 direction. The scalar components of the M_i^j multiplets describe the fluctuations of the N_f flavor D3-branes in the x^8 direction which is common to the D5 and $(1,N)$ branes. The dual quarks arise from the open strings stretching between the color and flavor D3-branes.

The dual theory possesses the tree-level superpotential interaction

$$\int d^2\theta \left(\tilde{q}^i \widetilde{\Phi}_3 q_i + M_i^j \tilde{q}^i q_j\right) + c.c. \ . \tag{4.3}$$

A short explanation of this interaction proceeds in the following way. The flavor D3-branes are stuck at $x^3 = 0$. By moving the color D3-branes in the x^3 direction the dual quarks become massive, a fact which is captured by the first cubic interaction in the above superpotential. Similarly, the color D3-branes are stuck at $x^8 = 0$, whereas the flavor D3-branes can move in the x^8 direction making again the dual quarks massive. This fact is captured by the second cubic interaction in (4.3).

4.2 Adding an adjoint superfield with a tree-level superpotential

Another interesting generalization involves taking a general number n of NS5-branes in the brane configuration of figure 3. For simplicity, we set the number of D5-branes N_f to zero, but analogous statements can be made in the more general case. Then, the low-energy theory on the k suspended D3-branes is a U(k) $\mathcal{N}=1$ CS theory at level N coupled to two adjoint $\mathcal{N}=1$ superfields Φ and X. Once again, Φ is a massless multiplet whose lowest scalar component describes the fluctuations of the D3-branes in the x^3 direction. X is a multiplet whose lowest scalar component describes the fluctuations of the D3-branes in the x^8 direction. Since x^8 is not a common direction of the NS5 and $(1,N)$ branes the motion of the D3-branes along x^8 is not free. In the low-energy field theory on the D3-branes this effect is captured by a tree-level $\mathcal{N}=1$ superpotential $W(X)$ of degree $n+1$.

The precise form of W is closely related to the one-dimensional modulus of the brane setup that controls the x^8 position of the n NS5-branes. Placing the NS5-branes at n different points x_j^8, $j=1,\cdots,n$, forces the k D3-branes to break up into n groups of r_j D3-branes ending on the x_j^8 positioned NS5-brane with

$$\sum_{j=1}^{n} r_j = k \ . \tag{4.4}$$

From the D3-brane point of view x_j^8 are the real expectation values of the diagonal matrix elements of the scalar component of the superfield X. In field theory these vacua are accounted for by the $\mathcal{N}=1$ superpotential

$$W(X) = \sum_{j=1}^{n} \frac{s_j}{n+1-j} X^{n+1-j} \ . \tag{4.5}$$

For generic coefficient $\{s_j\}$ the superpotential has n distinct minima $\{x_j^8\}$ related to $\{s_j\}$ via the relation

$$W'(x) = \sum_{j=0}^{n} s_j x^{n-j} = s_0 \prod_{j=1}^{n} (x - x_j^8) \ . \tag{4.6}$$

The integers $(r_1, \cdots, r_n)$ label the number of the eigenvalues of the $N_c \times N_c$ matrix X residing in the j-th minimum (for r_j) of the scalar potential $V = |W'(x)|^2$. When all the expectation values x_j^8 are distinct the adjoint field is massive and the gauge group is Higgsed:

$$\mathrm{U}(k) \to \mathrm{U}(r_1) \times \cdots \times \mathrm{U}(r_n) \ . \tag{4.7}$$

In this vacuum we recover n decoupled copies of the $\mathcal{N} = 1$ AV CSM theory at level N.

At the origin of the NS5-brane moduli space (all $x_j^8 = 0$) the tree-level superpotential is $W(X) \sim \mathrm{Tr}\, X^{n+1}$. At weak coupling, $k/N \ll 1$, this is an irrelevant operator (for $n > 3$) that does not affect the IR dynamics. We will see in a moment that this is not true for sufficiently large coupling. At the same time quantum corrections generate a potential for both Φ and X.

From the s-rule of brane dynamics we learn that this theory has a supersymmetric vacuum if and only if

$$k \leqslant nN \ . \tag{4.8}$$

We also learn, by exchanging the fivebranes, that there is a Seiberg dual description in terms of a $\mathrm{U}(nN-k)$ $\mathcal{N} = 1$ CS theory at level N coupled again to two $\mathcal{N} = 1$ multiplets $\widetilde{\Phi}, \widetilde{X}$ in the adjoint.

As in the closely related $\mathcal{N} = 2$ examples of [6], the s-rule and Seiberg duality reveal some of the non-trivial properties of this theory. In particular, we learn that by increasing the large-N coupling $\lambda = k/N$ there is a point $\lambda_{n+1}^{\mathrm{SUSY}} = n$ where supersymmetry gets spontaneously broken in the presence of the tree-level deformation $\mathrm{Tr}\, X^{n+1}$. This implies that there is a critical coupling $\lambda_{n+1}^* < n$ beyond which the operator $\mathrm{Tr}\, X^{n+1}$ becomes relevant and affects the infrared dynamics. The Seiberg dual theory, which is weakly coupled when k/N is close to n, implies similarly that there is an upper value λ_{n+1}^{**} for k/N above which the IR theory is again unaffected by the $\mathrm{Tr}\, X^{n+1}$ deformation.

The picture that seems to be emerging from this information is the following. At weak coupling the dynamics of the $\mathrm{U}(k)$ theory at level N is controlled by the loop-generated effects and is described at low energies by the pure CS theory. Duality relates this topological theory to a strongly coupled $\mathrm{U}(nN-k)$ theory with a relevant $\mathrm{Tr}\, X^{n+1}$ deformation.

As we further increase the coupling we encounter a regime of parameters (in the large-N limit this regime is given by $k/N \in [\lambda_{n+1}^*, \lambda_{n+1}^{**}]$), where the theory flows in the infrared to an interacting fixed point. This conformal window appears in a non-perturbative region of the theory where both the $\mathrm{U}(k)$ and $\mathrm{U}(nN-k)$ descriptions of the theory are simultaneously strongly coupled. Notice that in postulating this conformal window we have assumed the inequality $\lambda_{n+1}^* < \lambda_{n+1}^{**}$. If the opposite inequality were true, we would have obtained an inconsistency. Inside the range $[\lambda_{n+1}^{**}, \lambda_{n+1}^*]$ the loop-generated mass terms would dominate the IR dynamics and both the $\mathrm{U}(k)$ and $\mathrm{U}(nN-k)$ theories would be described at low energies by the pure CS theories. Seiberg duality would then reduce to level-rank duality in the corresponding WZW models giving a result that is inconsistent with the n-dependent exchange $k \to nN - k$ provided by string theory.

At even larger coupling, the $\mathrm{U}(k)$ theory is strongly coupled with a $\mathrm{Tr}\, X^{n+1}$ deformation, but Seiberg duality provides a dual description in terms of a $\mathrm{U}(nN-k)$ theory where the tree-level deformation is irrelevant and the IR theory is again controlled by the pure CS Lagrangian.

A more complete analysis of the dynamics of this theory with a verification of the above scenario would be of interest.

5 Conclusions

In this paper we studied a class of $\mathcal{N} = 1$ supersymmetric Yang-Mills Chern-Simons theories in three dimensions capturing the IR worldvolume dynamics of domain walls in the four-dimensional SU(N) $\mathcal{N} = 1$ SYM theory and some of its generalizations. We argued that this class of CS theories can be reproduced as the low-energy dynamics on the worldvolume of D3-branes suspended between an NS5-brane and a $(1, N)$ fivebrane bound state in a type IIB string theory setup. By T-duality this setup resembles the type IIA large-N holographic dual of $\mathcal{N} = 1$ SYM theory presented in [22], but is not identical to it.

The brane configuration in type IIB string theory provides a simple intuitive understanding of some of the most characteristic properties of the $\mathcal{N} = 1$ SYM domain walls and the CSM theory that lives on them. These properties include: the identification of the $N - 1$ BPS domain walls with supersymmetric D3-brane configurations, a geometric counting of the Witten index and the degeneracy of domain walls, and a strong/weak coupling Seiberg-type duality of the CS worldvolume theory that relates the k-th wall with the $(N - k)$-th anti-wall.

The brane construction is also a promising route for several generalizations. Two of them were considered in section 4. By adding D5-branes one adds $\mathcal{N} = 2$ flavor multiplets to the matter content of the AV theory; by adding more NS5-branes one adds an $\mathcal{N} = 1$ scalar multiplet in the adjoint with a tree-level superpotential. It would be interesting to obtain a more complete understanding of the IR dynamics of the resulting $\mathcal{N} = 1$ CSM theories. Moreover, it would be interesting to know if these additional ingredients can also be embedded within a four-dimensional gauge theory context.

An interesting open problem is to generalize the Acharya-Vafa theory to the case of $\mathcal{N} = 1$ SYM with SO (or Sp) gauge group. For this one needs to consider a generalization of the brane setups in this paper that includes the appropriate orientifolds. The resulting three-dimensional worldvolume theory on the domain walls of these theories should admit again a Seiberg duality which relates the k-th wall with the $(N - k - 2)$-th wall (or $(N - k + 2)$-th wall).

Another generalization that presumably involves orientifolds is related to the domain wall worldvolume theory of a four-dimensional SU(N) gauge theory with a Dirac fermion in the two-index anti-symmetric (or symmetric) representation. Despite being a non-supersymmetric theory we expect $N - 2$ (or $N + 2$) degenerate vacua, since the axial U(1) is broken by the anomaly to $\mathbb{Z}_{2(N-2)}$ (or $\mathbb{Z}_{2(N+2)}$) and then further to $\mathbb{Z}_2$. Moreover, it has been argued [30, 31] that in the large-N limit this theory should admit 'BPS' domain walls. Therefore, we expect an Acharya-Vafa-like theory on its k-walls.

We conclude the paper by posing a question. We argued that the underlying reason for Seiberg duality in the three-dimensional worldvolume theory is a four-dimensional charge-conjugation symmetry. Is it possible that the original Seiberg duality for $\mathcal{N} = 1$ SQCD in four dimensions [32] is due to a discrete symmetry in a higher dimensional theory?

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